Teachers’ use of computational tools to construct and explore
dynamic mathematical models

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To what extent does the use of computational tools offer teachers the possibility of constructing dynamic models to identify and explore diverse mathematical relations? What ways of reasoning or thinking about the problems emerge during the model construction process that involves the use of the tools? These research questions guided the development of a study that led us to document the process exhibited by high school teachers to model mathematical situations dynamically. In particular, there is evidence that the use of computational tools helped them identify and explore a set of mathematical relations or conjectures that appear throughout the interaction with the task. Thus, the participants had the opportunity of fostering an inquisitive approach to the process of model construction that led them to formulate conjectures or mathematical relations and to search for ways to support them.

Keywords: dynamic modelling; technology; high school teachers

1. Introduction

Modelling activities play an important role in developing mathematical comprehension. Swan et al. [1] identify several projects in which a model approach is central to foster students’ development of mathematical learning. They pointed out that ‘modelling is a powerful promoter of meaning and understanding in mathematics. When presented with problems set in some real world context, students formulate questions about the context and think about the usefulness of their mathematical knowledge to investigate questions’ (p. 281). Similarly, Kaiser et al. [2] mention... ‘Together with the use of information technology, the introduction of mathematical modeling and applications is a prominent general feature of the recent developments in the practice of mathematics teaching’ (p. 82). In particular, the modelling cycle that involves examining the phenomenon to be modelled, identifying and discussing assumptions and conditions to construct the model, and constructing, exploring, and validating the model provides useful information to frame an instructional approach to guide teachers’ practices that aim to foster their students’
mathematical thinking. Maaß [3] cited the work of Blum and Kaiser [4] to refer to modelling competences that are involved in the modelling process. These competences appear and are relevant during different phases of the modelling cycle and they include: comprehension of the phenomenon (making assumptions, recognizing parameters and quantities, identifying relations); construction of a model (mathematizing the phenomenon, simplifying relations and quantities, choosing proper notation); exploring and responding to questions (using heuristic methods and mathematical concepts); interpreting results (validating, questioning, and generalizing solutions).

In this context, we argue that a central activity in students’ process of developing mathematical concepts and solving problems is the construction of models that are used to identify, explore, and support mathematical relations. Goldin [5, p. 184] states that ‘... A model is a specific structure of some kind that embodies features of an object, a situation or a class of situations or phenomena – that which the model represents’. How is a model constructed? How can one evaluate the pertinence of a model? What is the role of the use of computational tools in the construction of models? To respond to and discuss these questions, we identify an inquisitive or inquiring approach as a crucial activity associated with the modelling process.

Mathematical modelling is the process of encountering an indeterminate situation, problematizing it, and bringing inquiry, reasoning, and mathematical structures to bear to transform the situation. The modelling produces an outcome – a model – which is a description or a representation of the situation, drawn from the mathematical disciplines, in relation to the person’s experience, which itself had changed through the modelling process (Confrey and Maloney [6, p. 60]).

Sriraman and Lesh [7, pp. 247–248] argue that during the study of mathematics and science, it is crucial for teachers and students to focus on modelling activities because:

1. Modelling is primarily about purposeful description, explanation, or conceptualization (quantification, dimensionalization, coordinatization, or in general mathematization) – even though computation and deduction processes also are involved.
2. Models for designing or making sense of complex systems are, in themselves, important ‘pieces of knowledge’ that should be emphasized in teaching and learning – especially for students preparing for success in future-oriented fields that are heavy users of mathematics, science, and technology. Therefore, it is important to initiate and study modelling, particularly those of complex systems that occur in real life situations from the very early grades.

What type of ‘complex system’ should teachers think of in order to foster their students’ model construction of different phenomena? What does the modelling process involve? Lingefjärd [8] mentioned that: ‘[m]athematical modeling can be defined as a mathematical process that involves observing a phenomenon, conjecturing relationships, applying mathematical analysis (equations, symbolic structures, etc.), obtaining mathematical results, and interpreting the model’ (p. 96). Teachers need to problematize their instructional practice in order to design instructional routes. The construction of an instructional route that emphasizes modelling activities demand that teachers themselves recognize that modelling activities provide an opportunity for their students to develop and use mathematical ideas.
To this end, it is crucial for teachers to engage into an inquisitive approach to examine the situation or phenomenon (formulation and discussion of questions) in terms of mathematical resources and strategies that lead them to the construction of models [9]. A model then is a vehicle for teachers to identify mathematical relations and to solve problems. We argue that the development and availability of computational tools offer teachers the possibility of enhancing their repertoire of resources and heuristics strategies to deal with mathematical relations embedded in models. It is also important to recognize that different tools may offer distinct opportunities for them to represent and approach mathematical problems. For example, with the use of dynamic software, such as Cabri-Geometry, Sketchpad, or Geogebra, some tasks can be modelled dynamically as a means to identify and explore diverse mathematical relations or conjectures. Thus, tasks or problems are seen as opportunities for teachers and students to engage in the construction of models. In this context, they pose and pursue relevant questions that lead them to identify and represent relevant information that guides that construction. In this study, high school teachers worked on a series of mathematical tasks in which they had the opportunity to construct and explore mathematical models. Those models provided them relevant information to think of and design their instructional routes. They were encouraged to use dynamic software during the process of constructing and refining the models.

2. Conceptual foundations

Kelly and Lesh [10] have recognized that researchers, teachers, and students rely on models to represent, organize, examine, and explain situations. For instance, researchers construct models to analyse and interpret teachers and students’ activities. Teachers use models to describe, examine, and predict students’ mathematical behaviours, while students use models to describe, explain, justify, and refine their ways of thinking. Thus, a model is conceived of as a conceptual unit or entity to foster and document both the teachers’ construction of instructional routes and the students’ development of mathematical knowledge. As Burkhardt [11] stated, ‘learning to model with mathematics is at the heart of learning to do the analysis that guides understanding and sensible decisions’ (p. 180).

In this context, it becomes important to identify not only the basic ingredients or elements of a model; but also to characterize the process involved in the construction of models. Lesh and Doerr [12] state that a model is a system formed by elements, relationships among the elements, operations and rules or patterns to justify the use of the operations. They also mention that: ‘to be a model, a system must be used to describe some other system, or to think about it. Also to be a mathematically significant model, it must focus on underlying structural characteristics of the system being described’ (p. 362). That is, the model construction involves examining the situation or problem to be modelled in order to identify essential elements that need to be represented and scrutinized through operations and rules with the aim of identifying and exploring mathematical relations [13]. Mathematical models also need to be contrasted and refined. That is, ‘models evolve by being sorted out, refined, or reorganized at least often as they evolve by being assembled (or constructed)’ [12, p. 365]. Here, we are interested in documenting cognitive behaviours that high school teachers exhibit during the interaction with the task.
Thus, it is important to distinguish phases or cycles that explain relevant moments around the teachers’ or students’ process involved in the construction of models. In particular, ways in which teachers refine or transform initial models of the situation or phenomenon into more robust or improved models to deal with the situation.

In order to identify the essential elements embedded in a task or phenomenon, it is important to comprehend initially the situation or the problem [14]. Understanding phenomena or situations that involve real contexts demands not only the recognition of the key elements around the problem; but also ways to represent them mathematically. This phase is crucial to think of a way to construct the model of the problem that will be explored through mathematical resources and strategies. This stage requires that teachers identify and rely on a set of conditions or assumptions that will be used to construct the model. The model exploration stage leads the problem solver to search for different approaches and media to examine the model and eventually to solve the problem. The next stage is to interpret and validate the solution in terms of the original statement or problem conditions. In this process, it is important to analyse and discuss whether the model used to solve the problem or situation represents a tool to approach a family of problems or situations. Thus, questions that can guide the model construction process involve: Is the model of the situation appropriate? Is the solution reasonable and consistent with the problem statement? Can the model be improved? Can the model be extended? What are the mathematical resources, concepts and strategies that were relevant during the construction and exploration of the model? These types of questions are crucial to evaluate the strengths and limitations of the model and to extend the model scope [15,16]. Figure 1 shows elements and actions associated with the modelling cycle where real world phenomena are examined in terms of assumptions and data to be represented and explored through mathematical resources and operations.

Figure 1. Modelling cycle.
Later, results are interpreted and validated in relation to the original context and condition of the problem. As a consequence, the model might be refined or extended.

How can the use of computational tools influence and help the problem solver construct and explore mathematical models? Recently, several research programmes have analysed and documented the role played by the use of diverse digital tools in the students’ learning of mathematical knowledge [17]. For example, Sacristan et al. [18] document that the use of dynamic software offers advantages to construct models of situations or problems in which the models’ components can be displaced within the representation to identify and explore mathematical relations; while the use of calculators offers advantages to represent and deal with problems algebraically. It is also recognized that for teachers or learners to use a tool efficiently, they need to be engaged into a tool appropriation process where they gradually transform an artefact into an instrument. This tool appropriation depends on cognitive schemata that learners develop during the use of the tool to represent and explore the problem. Drijvers et al. [19] pointed out that:

The process of an artifact becoming part of an instrument in the hands of a user…is called instrumental genesis…Instrumental genesis is an ongoing, nontrivial and time-consuming evolution A bilateral relationship between the artifact and the user is established: while the student’s knowledge guides the way the tool is used and in a sense shapes the tool (this is called instrumentalization), the affordances and constraints of the tool influence the student’s problem solving strategies and the corresponding emergent conceptions (this is called instrumentation) (pp. 108–109).

Trouche [20, p. 285] stated that, ‘an instrument can be considered an extension of the body, a functional organ made up of an artefact component (an artefact, or the part of an artefact mobilized in the activity) and a psychological component’. That is, the artefact characteristics (ergonomics and constrains) and the schemata developed by the students during the activities are important for them to transform the artefact into a problem-solving instrument. In this respect, Trouche [20, p. 286] related the students’ psychological component to construction of a scheme with three functions: ‘a pragmatic function (it allows the agent to do something), a heuristic function (it allows the agent to anticipate and plan actions), and an epistemic function (it allows the agent to understand what he is doing)’. As a consequence, the process of model construction and exploration also incorporates a variety of ways to represent, formulate, and examine mathematical relations. For instance, with the use of the tool, the situations or problems are now analysed in terms of the facilities and affordances offered by the tool, such as dragging particular components; finding loci of points or lines, quantifying certain relations, using a Cartesian system to model the problem algebraically [21]. Indeed, these three functions become essential to construct dynamic models of problems. As a consequence, the same process of model construction and exploration incorporates new ways to represent, formulate, and explore mathematical relations. For example, the situations or problems are now analysed in terms of the facilities offered by the tool such as dragging particular components; finding loci of points or lines, quantifying certain relations, and using a Cartesian system to algebraically model the problem [21]. Indeed, the use of the tool offers the problem solvers the opportunity of exploring new routes to develop or reconstruct and explore basic mathematical results. In particular, the visual approach becomes relevant to identify mathematical relations that later can be analysed in terms of numeric and graphic approaches.
The research questions used to guide and structure the development of the study were: (1) what ways of reasoning or thinking about the problems emerge during the model construction with the use of the tools?, (2) to what extent does the use of computational tools offer teachers the possibility of constructing dynamic models of problems to identify and explore diverse mathematical relations?, and (3) what types of mathematical resources and strategies emerge during the teachers’ construction of mathematical models associated with the phenomena?

3. Research design, methods, and general procedures
Six high school teachers participated in 3 h-weekly problem-solving sessions during one semester. These teachers were involved in a masters programme in mathematics education and they had experiences in using computational tools like dynamic software and hand-held calculators. All had completed their BS with major in mathematics or engineering which is required to become a high school teacher. During the semester, there were 20 sessions and the task analysed in this article was discussed during three sessions. Throughout the sessions, the teachers were encouraged to use dynamic software (Cabri Geometry) to construct dynamic models associated with the tasks or situations.

In general, the didactic approach during the sessions involved working in pairs and plenary presentations. Two researchers coordinated the development of the sessions and participated as members of a community that fostered an inquisitive approach to the tasks. In this process, the participants (including both researchers) worked as a part of the community not only to solve the problems; but also to have opportunities to review mathematical content that emerged while solving the tasks. The problem-solving sessions were recorded and each pair handed in a report that included the software files. The researchers took notes and discussed the advantages of using the tool during the diverse problem-solving phases. We focus on analysing the work shown by the community while dealing with a problem embedded in a real context (the church view task).

It is important to mention that the unit of analysis is the work shown by the six participants as a group during the sessions. The task discussed throughout this study is representative of the type of problems that the participants addressed during the development of the sessions.

3.1. The task
Figure 2 shows a car going on a straight roadway. Aside, there is an old church and the driver wants to stop so that his friend (the passenger) can appreciate the façade of the church. At what position of the roadway should the driver stop the car, so that his friend can have the best view? (Adjusted from [22]).

Lesh and Doerr [23] recognize the relevance for students to focus on model construction activities to reveal, develop, and refine their mathematical ideas and problem-solving proficiency.

... model-eliciting activities—so called because the products that students produce go beyond short answers to narrowly specified questions— which involve sharable, manipulatable, modifiable, and reusable conceptual tools (e.g., models) for constructing, describing, explaining, manipulating, predicting, or controlling mathematically
significant systems. Thus, these descriptions, explanations, and constructions are not simply processes that students use on the way to producing ‘the answer;’ and, they are not simply postscripts that students give after “the answer” has been produced. They ARE the most important components of the responses that are needed. So, the process is the product!. [23, p. 23].

We argue that the church task used in this study shares main properties associated with model-eliciting activities. For example, the construction of a dynamic model of the task relies on identifying and relating key information of the task to software commands; and the exploration of the model not only becomes a vehicle to elicit the problem solver’s ideas, but also a source to generate mathematical relations and ways to support them.

4. Presentation of results

We organize and structure the presentation of results in terms of identifying essential phases around the process of model construction that the participants exhibited during the interaction with the task. This structure is consistent with modelling cycles described in the conceptual foundations of the study. The analysis focuses on characterizing the ways of reasoning shown by the participants during the modelling process of the task. We emphasize how the use of the tool shaped not only the way to think of the modelling process, but also to support results. The phases include the initial comprehension of the statement of the task; the identification of basic elements and conditions to construct a model of the problem; the exploration of the
model and the formulation of conjectures; and ways to support mathematical relations and conjectures, interpretation of results, validation, and extension of the model. Table 1 gives a description of main activities that the participants addressed during the development of the three sessions in which the task was discussed. Later, we identify and discuss relevant mathematical features that the participants showed during their interaction process.

### 4.1. Understanding the task statement: an inquiring approach

This phase was important to comprehend the task and to identify and discuss a set of assumptions that led the group to identify the elements to be considered in the model construction. To this end, the group posed and discussed several questions that included: How can we identify that for a distinct position of the car, the passenger has different views of the church? What does it mean to have the best view? Is it sufficient to consider the position of the passenger on the roadway as a reference instead of the car to determine the best position? It was observed that the figure provided in the statement of the task helped them to visualize and eventually represent the problem. They relied on the figure to assume that the observer could be identified as one point that is moved along a line (the roadway). Here, the participants also discussed if the provided information was sufficient to solve the task since there was no quantitative data involved in the statement.

Emily: I see that the church façade is not parallel to the highway and it can be represented through a segment

<table>
<thead>
<tr>
<th>Session</th>
<th>Modelling activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Understanding the task, identification of relevant information, representing the information through mathematical objects, constructing a dynamic model, quantification of parameters, and visualizing their behaviours. Formulation of initial conjectures.</td>
</tr>
<tr>
<td>2</td>
<td>Systematic exploration of the initial conjecture that relates the best view of the church to the tangent circle to a line (roadway) that passes through the end points of the view. Focusing on empirical and visual approaches.</td>
</tr>
<tr>
<td>3</td>
<td>Looking for geometric arguments to support the model and to examine other relations that enhance the initial model of the problem.</td>
</tr>
</tbody>
</table>

Table 1. Main activities addressed during the three problem-solving sessions.
Initially, the participants identified two ways to characterize the best view of the façade: one that related the distance between the observer and the façade (less distance better view) and the other that focused on relating the best view to the angle that is formed between two points on the façade and the observer. The former interpretation was chosen by the participants and became a source to construct the model of the situation. At this moment, the task was thought of in terms of the basic elements (lines, angles, segments, etc.) as a way to construct a mathematical model.

4.1.1. Comment

This first comprehension stage was important for the participants to agree on a series of assumptions to relate the façade view to the position of the observer and to focus on the angle variation. That is, the angle formed by a point (vertex), (the observer), and the segment (façade) end points.

4.1.1.1. Model construction. The initial analysis of the statement led the participants to construct a dynamic model of the situation by representing elements of the task (roadway, façade of the church, and observer) through geometric objects (lines, segments, points, angles). In this context, Sophia proposed to represent the roadway with a straight line and the passenger a point on that line, and a segment as the base of the church’s façade (Figure 3, right).

Daniel: Why can the façade of the church be represented through a segment?
Peter: . . . because the segment represents the wide of the church at the middle part (pointing to the picture).

Thus, the participants argued that the best view of the church could be identified by comparing the angle variation that relates the width of the church (represented by a segment) to the point represented by the observer. The participants agreed that Sophia’s representation included the relevant information of the task. It is important to mention that in order to explain what happens to the angle for various positions of the point (the observer), they recognized that it was necessary to identify and represent explicitly the main objects embedded in the problem (points, angles, line, segment). Here, some participants initially used paper and pencil to sketch a problem.
representation but sooner they recognized that the use of the tool (dynamic software) provided a means to visualize the angle variation for different positions of the observer. In this model, the point \( P \) on a line \( L \) represented the observer (Figure 3, right).

4.1.1.2. Model exploration and conjectures. At this stage, there appeared two ways to represent the statement: One in which some participants used paper and pencil and relied on trigonometric relations to construct and explore the model (Figure 3, left); and the second approach in which the use of the tool guided the model exploration. Thus, the participants who decided to use the software, started to observe the behaviour and relations between angle \( APB \) and attributes of some objects within the configuration (lengths, areas, and perimeters) when point \( P \) was moved along line \( L \) (Figure 4).

In particular, Sophia and Jacob noticed that when point \( P \) was situated on a position that was collinear with points \( A \) and \( B \), then the angle \( APB \) measured zero degrees; but when point \( P \) was moved to the left of the collinear position,
the measurement of angle \( APB \) increased for some positions and then the angle value decreased. Thus, they recognized that there was a position for point \( P \) where the angle value reached its maximum value; and they focused on determining the position for point \( P \) on \( L \) to identify the maximum value of angle \( APB \).

There appeared two empirical ways to identify the maximum value of the angle:

1. One in which some of the participants directly visualize the numbers displayed in the table while moving point \( P \) along the line \( L \) (Figure 4); and,
2. Another way in which the participants constructed a graphic representation that involves the distance \( AP \) and the corresponding value of angle \( APB \) (Figure 5).

4.1.2. Comment

With the help of the software, the participants had the opportunity to quantify directly the distance \( AP \) and the angle measure in a table to observe their behaviours when \( P \) is moved along line \( L \). In addition, they could graph the function that relates distance \( AP \) to the angles values without having an algebraic representation. At this stage, the dynamic model provided an empirical and graphical solution to the problem.

During the group discussion, the participants recognized that it was important to look for an algebraic or geometric argument to justify the position of point \( P \) where the angle reaches its maximum value. In this process, Daniel and Emily decided to draw the circle that passes through points \( P, A, \) and \( B \). Based on this construction, they realized that when point \( P \) is moved along line \( L \), then the circle that passes through points \( P, A, \) and \( B \) seems to be tangent to line \( L \) at the position where angle \( APB \) reaches its maximum value (Figure 6).
A conjecture emerged: To identify the point where angle $\angle APB$ reaches its maximum value is sufficient to draw a tangent circle to line $L$ that passes through points $A$ and $B$. That is, the tangency point of the circle and line $L$ is the place where the observer gets the best view of the church. How can we construct the circle that passes through $A$ and $B$ and is tangent to line $L$? Emily posed this question to the rest of the participants during the class discussion.

Sophia and Jacob suggested that in order to identify relevant properties of the tangent circle they could assume its existence. That is, if the tangent circle exists, what properties should it have? Here, it was recognized that the circle must lie on the perpendicular bisector of segment $AB$ and also that its centre must also be on the perpendicular to line $L$ that passes through the tangency point. Thus, Daniel drew a perpendicular line to $L$ that passed through point $P$ and the perpendicular bisector of segment $AP$. These lines intersected at point $D$; and they asked: What is the locus of point $D$ when point $P$ is moved along line $L$? With the use of the software, the locus was determined (Figure 7). Thus, the intersection point ($C$) of the locus and the perpendicular bisector of segment $AB$ was the centre of the tangent circle. Here, to draw the circle, they drew the perpendicular from point $C$ to line $L$, and the distance from point $C$ to line $L$ was the radius of the tangent circle (Figure 7). During the session, it was also argued that the locus of point $D$ when point $P$ is moved along line $L$ is a parabola, since point $D$ is on the perpendicular bisector of segment $AP$ and it holds that $d(P, D)$ is always the same as $d(D, A)$; definition of perpendicular bisector. Here, the focus of the parabola is point $A$ and the directrix is the line $L$.

4.1.3. Comment

A key issue to draw the tangent circle to line $L$ was to assume its existence (Polya’s heuristic). This led the group to focus on drawing the perpendicular bisector of segment $AB$, a perpendicular line to $L$ that passes through the tangency point $P$, and others.
and constructing the perpendicular bisector of segment $AP$. By moving point $P$, a parabola was generated and this provided the information needed to draw the tangent circle. Thus, this dynamic model facilitated finding and using relations to draw the tangent circle.

An algebraic argument to show that the locus of point $D$ was a parabola was provided by the participants during the group discussion. The argument involves representing point $P$ with coordinates $(t, 0)$ and point $A(a, 0)$. Thus, the slope of the line that passes through points $A$ and $P$ is equal to $-a/t$ and the slope of the perpendicular bisector $AP$ is equal to $a/t$, and therefore, the equation of the perpendicular bisector is $y = (t/a)(x - t/2) + a/2$. Besides, the coordinates of point $D$ can be found by solving the system:

$$\begin{cases} y = \frac{t}{a}(x - \frac{t}{2}) + \frac{a}{2} \\ x = t \end{cases}$$

Then, the coordinates of point $D$ are $(t, t^2/2a + a/2)$, that is, point $D$ is moved along the parabola with vertex at point $(0, a/2)$, and focus $(0, a)$, and directrix $y = 0$.

Daniel and Emily constructed the tangent circle to line $L$ that passes through points $A$ and $B$ by drawing initially the perpendicular bisector of segment $AB$. Later, they situated a point $C$ on that perpendicular bisector and drew a circle with centre point $C$ and radius the segment $CA$. They also drew a perpendicular to line $L$ that passes through point $C$. This perpendicular and the circle get intersected at point $D$. What is the locus of point $D$ when point $C$ is moved along the perpendicular bisector of $AB$?

Again, the use of the software showed that such locus was one branch of a hyperbola. The locus intersects line $L$ at point $P$. The perpendicular line to $L$ that passes through point $P$ intersects the perpendicular bisector of segment $AB$ at point $C'$. Thus, to draw the tangent circle to $L$ that passes through points $A$ and $B$, it was sufficient to draw the circle with centre point $C'$ and radius segment $C'P$ (Figure 8).
4.2. An algebraic approach

Peter and Paul decided to approach the problem algebraically. They used the Cartesian system to situate the main elements of the problem (Figure 9).

Their initial goal was to determine the lengths of the sides of triangle $ABP$. That is,

$$AB = \sqrt{(m - r)^2 + (n - s)^2}$$

$$AP = \sqrt{(m - x)^2 + n^2}$$

$$BP = \sqrt{(r - x)^2 + s^2}$$
Then, by using the cosine law they found that:
\[(AB)^2 = (AP)^2 + (BP)^2 - 2(AP)(BP) \cos \theta,\]
and this expression can be written as:
\[
(m - r)^2 + (n - s)^2 = [(m - x)^2 + n^2] + [(r - x)^2 + s^2] - 2\sqrt{(m - x)^2 + n^2}\sqrt{(r - x)^2 + s^2} \cos \theta
\]

By isolating \(\theta\), the expression becomes a function of the coordinate \(x\) of point \(P\). That is,
\[
\theta = \arccos \left[ \frac{[(m - x)^2 + n^2] + [(r - x)^2 + s^2] - (m - r)^2 - (n - s)^2}{2\sqrt{(m - x)^2 + n^2}\sqrt{(r - x)^2 + s^2}} \right]
\]
\[
= \arccos \left[ \frac{x^2 - (m + r)x + (mr + ns)}{\sqrt{(m - x)^2 + n^2}\sqrt{(r - x)^2 + s^2}} \right]
\]

At this stage, they used the Derive software to find the derivative and the points where the derivative is equal to zero (Figure 10).

Based on previous result, they observed that the maximum value of the function is when:
\[
x = \frac{\sqrt{ns\sqrt{m^2 - 2mr + n^2 - 2ns + r^2 + s^2} + ms - nr}}{s - n}
\]
By substituting particular values for \( m, n, r, \) and \( s \), they constructed the associated dynamic representation and observed that then angle maximum is approximately equal to \( 58.74^\circ \) and this value is obtained when the coordinate of point \( P \) is 3.2 (Figure 11).

4.2.1. Comment
An important problem-solving strategy to analyse the model algebraically is to represent the model components in such a way that it was easy to deal with the operations involved. For example, the participants assumed (without losing generality) that the roadway could be represented as the \( x \)-axis. The use of the Derive software also helped them to carry out the operations involved in the process to construct the algebraic model.

4.2.2. Connections
When the participants focused their attention to finding a position for point \( P \) where the circle that passes through the end points of segment \( AB \), they had the opportunity to examine properties of tangents circles to a line. In that context, one of the participants wondering whether a well-known result that involved a tangent and a secant line to a circle was related to the problem. This question eventually led the group to introduce another approach to the problem.

4.2.3. The secant theorem
Another way to solve the problem was to assume that the problem was solved and by observing mathematical properties associated with the solution. Figure 12 shows a possible solution to the problem.

By observing the figure, the participants recalled a theorem: if \( Q \) is an outside point of a circle and from \( Q \) a tangent and secant lines are drawn to the circle, then if
the intersection points of the secant line and the circle are \( A \) and \( B \) and the tangency point is \( P \), the relation \( QP^2 = QA \cdot QB \) holds.

In this case, the known data include distance \( QA \) and distance \( QB \) (Figure 13, left) and with this information it is possible to determine the distance between \( Q \) and \( P \) by using the expression \( QP^2 = QA \cdot QB \). An interpretation of this expression is that segment \( QP \) is the proportional mean of segments \( QA \) and \( QB \). Thus, to solve the problem they followed the following steps:

1. Draw point \( B' \) that is symmetric to point \( B \) with respect point \( Q \).
2. Draw the circle with centre point \( M \) (middle point of segment \( B'A \)) and radius \( d(M, A) \).
3. Draw a perpendicular to line \( AB \) that passes by \( Q \). This line cuts the circle at point \( D \). The segment \( DQ \) is the mean proportional of \( QA \) and \( QB \) because the rights triangles \( B'QD \) and \( DQA \) are similar (Figure 13, right).

In triangles \( B'QD \) and \( B'DA \), it is observed that \( \angle B'DQ = 90 - \angle QB'D \) (triangle \( B'QD \)); in addition, \( \angle DAQ = \angle DAB' = 90 - \angle QB'D \) (triangle \( B'DA \)); therefore, \( \angle B'DQ = \angle DAQ \) and \( \angle QB'D = QDA \). Therefore, \( DQ \) is the mean proportional of \( QA \) and \( QB \).

4. Draw a circle with centre at point \( Q \) and radius \( QD \), this circle intersects line \( L \) at points \( P \) and \( P' \). The centres of the circles that are tangents to line \( L \) and
that pass through points $A$ and $B$ are the intersection points ($C$ and $C'$) of the perpendicular bisector of segment $AB$ and the perpendiculars to $L$ that pass through point $P$ and $P'$, respectively (Figure 14).

4.2.3.1. Interpretation and model validation. When the participants began to make sense of the task statement, they questioned the extent to which the car passenger would actually search for a place to stop the car to get the best view of the old church; however, they also recognized that the set of assumptions that they identified and used to model the task were consistent and could correspond to one possible interpretation of the statement. It is important to mention that this task was approached after they had addressed other similar tasks in previous sessions. These experiences might have helped them think of a model that reduces the complexity involved in the statement picture (Figure 2). When they accepted the model proposed by Sophia (Figure 3, right) then their model exploration led them initially to identify visually and graphically the maximum angle value. Later, they observed that the model included a triangle $ABP$ and drew circle that passes through the three vertices of that triangle. When they moved point $P$ along line $L$, they observed that there was a relationship between the maximum angle value and the tangent circle to line $L$. At this stage, all the participants focused their attention to exploring properties and emerging relations associated with the dynamic model of the situation. For example, during the class discussion, the participants recognized that the problem of finding the best view was reduced to construction of a tangent circle to a line that passes through two given points. Here, they all recognized that it was important to provide a mathematical argument to validate that the tangency point was the position where the angle gets its maximum value. To present the argument, they relied...
on Figure 15: M and N are the intersection points of the perpendicular bisector of segment AB and circles that pass through points ABD and ABP₀, respectively.

Thus, to compare the values of angle ADB and angle AP₀B is the same as comparing angles AMB and ANB. This is because angle ADB is congruent with angle AMB and angle ANB is congruent with angle AP₀B. It is also observed that d(A, N) becomes equal to AP when D coincides with point P (tangency point), otherwise, d(A, N) is always larger than AM. Therefore, the tangency angle is the angle with maximum value.

To evaluate the appropriateness and feasibility of the model, the participants changed the original position of the essential elements (roadway and façade) and they observed that the model also allowed them to identify the best view of the façade. Including the case in which the façade (segment AB) and the roadway (line L) are parallel, here the best view appears at the intersection of the perpendicular bisector of segment AB and line L.

The participants also observed that the domain of the solutions lies on the interval between the intersection of the perpendicular from the extreme of the façade that is closest to the line (roadway) and the intersection point of the perpendicular bisector of segment AB and the roadway (Figure 16). In terms of interpreting the model results and their relation with the original statement of the task, the participants argued that it was difficult for them to imagine an observer measuring an angle of 38° or 58° to focus on a part of the church. Some of them suggested that they could think of a camera with angle view fixed to that value and to move that camera along a line to fit that angle view with the ending point of a segment drawn on the church.

5. Discussion and remarks

It is relevant to reflect on key issues that appeared during the development of the sessions that focused on model construction and exploration. For example, it was
evident that the modelling approach used to guide the development of the problem-solving sessions helped the participants to focus on key aspects associated with the development of mathematical thinking and practice. That is, the task triggered the participants’ curiosity to engage them in a model construction process. The National Council of Teachers of Mathematics (NCTM) [24] recognized the relevance for learners to make sense of problem statements and to develop different ways to represent and reasoning about those problems. In this context, the participants realized that the process of initially comprehending the problem statements is crucial not only to identify essential aspects of the situation, but also to recognize a series of assumptions needed to construct a model of the task or problem. According to Polya [14], understanding means being able to make sense of the given information, to identify relevant concepts, and to think of possible representations to explore the problem mathematically.

In using the tool, the participants recognized that information embedded in the task needs to be thought in terms of mathematical objects and their properties in order to construct a dynamic model that can be explored by moving some elements within the model. That is, the use of the tool shapes the way to think of the model to represent the task [18]. Thus, during the model’s exploration phase, the participants had the opportunity to examine the model from distinct perspectives with the aim of identifying a set of relations or conjectures. At this stage, the participants used previous knowledge not only to identify those conjectures, but also to examine their plausibility. For example, by moving and observing the behaviour of a circumscribed circle in a triangle, they conjecture a relation between the tangent circle and the largest angle. Later, the conjectures that emerged, during the exploration phase, needed to be supported through mathematical arguments. Finally, the model used to solve the problem was examined in order to evaluate and contrast its pertinence and

Figure 16. The model’s domain.
possible extensions to be used in isomorphic or related tasks. According to Lesh and Doerr [23], the model needs to be reusable and in this case, the participants recognized that many tasks that involve variation of parameters could be modelled dynamically and as a consequence they can be explored through empirical, visual, and analytical approaches. Thus, the modelling cycles that involve those approaches were useful for the participants to examine and revise their own mathematical ideas. Lesh and Sriraman [25] pointed out that ‘in order to develop artefacts + designs that are sufficiently powerful, sharable, and reusable, it usually is necessary for designers to go through a series of design cycles in which trial products are iteratively tested and revised for specified purposes. Then, the development cycles automatically generate auditable trails of documentation which reveal significant information about the products that evolve (p. 492). In general terms, during the modelling cycles that the participants showed during the development of the sessions involved focusing on phases where the use of the tools became relevant to construct and explore the model:

1. Competence to construct a dynamic model. There is evidence that the use of the tool helped the participants to initially construct a dynamic model of the task. Thus, moving a point \( P \) on a line (roadway) led them to identify and relate the ‘best view’ with the angle formed between the ends of a segment (church façade) and that point. How can we measure the angle for distinct positions of point \( P \)? How can we identify the angle with a largest value? The participants used the software to measure the angle for various positions of point \( P \) to observe that there was a position where the angle’s value was the largest.

2. Competence to visually and graphically explore the model. The use of the tool became important to explore the model through visual and empirical approaches. For example, the graphic solution demanded that the participants thought of a functional approach in which the use of a Cartesian system helped them to relate the distance from one end of the segment \( AB \), the church façade and its corresponding angle. Thus, the graphic approach helped them visualize and identify the point where the angle reaches its maximum value. In addition, moving the point \( P \) on the line helped the participants to observe particular behaviours of other objects (circles, segments) within the representations.

3. Competence to formulate conjectures. Formally, the empirical and graphic approach to the problem provided basis for the participants to identify an approximated answer to the main question of the task. To this end, they observed that the circle that passes through the three points \( A, B, \) and \( P \) becomes tangent to the line when the angle \( APB \) reaches its maximum value. Thus, the solution of the task was reduced to draw a circle tangent to the line and the tangency point was the desired point. Later, this conjecture led them to identify and explore a series of mathematical relations.

4. Competence to use heuristic strategies. The use of the software helped the participants recognize and use a set of heuristics [14] (assuming the problem solved, relaxing certain conditions; finding loci of particular objects, and using the Cartesian system) to approach the problem. For example, they recognized that the locus of the intersection point of the perpendicular bisector of segment \( AB \) and the perpendicular line to \( L \) that passes through
point $P$, when $P$ is moved through line $L$ generated a parabola. Here, the parabola was the key to find the solution of the task.

(5) Competence to generate and use related mathematical results. The participants were surprised that a conic section was used to find the point where the observer gets the best view of the church. They also recognized that the dynamic representation of the problem became important to identify two mathematical results:

(1) Given a line $L$ and a segment $AB$ that is not parallel to line $L$; then there is a point $P'$ on the line where the circle that passes through points $A$, $B$, and $P'$ is tangent to line $L$. Here, point $P'$ is a point on line $L$ from which it is possible to observe segment $AB$ (the church façade) with the greatest possible angle $AP'B$.

(2) The locus of the centres of the tangent circles to a line $L$ and that pass through a given point is a parabola.

(6) Competence to search for different arguments to support results. The participants were aware of the relevance to think of different arguments to support conjectures or mathematical relations. In this process, they had the opportunity to think of the problem through diverse concepts that involves empiric, analytic, and formal approaches. To this end, they were able to relate concepts and theorems as the secant theorem to solving the task.

(7) Competence to use several tools. In this case, the use of dynamic software became important to model the problem; but also the use of Derive software helped them to deal with expressions involved in the algebraic approach. In this sense, the use of both tools is complementary.

Finally, the participants agreed that the process involved in the model construction of this task provided useful information for teachers to organize and structure their lessons. Such structure might involve an initial phase where students have the opportunity to construct a dynamic model of the task. This model then can be explored empirically and graphically through the use of the tool. As a result of that exploration, some conjectures or relations might emerge and they need to be supported in terms of geometric or algebraic arguments. Finally, the model and relations need to be interpreted in terms of the information provided in the original statement of the task. For example, the participants recognized that students might commence by analyzing key elements of the situation or problem to construct a dynamic model. This model can be explored visual and empirically by the students. This model exploration involves the identification of invariants, the quantification of parameters and relations, and the use of tables and graphs. The mathematical concepts that appeared during the exploration of the models provide the basis to think of and explore an algebraic model of the problem.

The construction of the dynamic representation, the quantification of attributes (measures of segments, angles, etc.), the identification of loci and the graphic representations are the key activities that can help students to identify and explore interesting mathematical relations. In addition, the use of the tools is also relevant to search for arguments to support those results. Thus, the dynamic model offered the participants the opportunity of visualizing a set of relations that became relevant to approach and extend the problem. In addition, the algebraic model not only became important to think of the problem in general terms; but also to use particular
heuristics (use of the Cartesian system, situate the roadway on the $x$-axis) and to relate the general solution to the solution achieved via the use of the software.

The use of Derive software made easy to operate the algebraic model to identify the maximum value of the function (point position and corresponding angle) through the use of calculus concepts. In this context, we argue that teachers and students should use various computational tools to represent and explore mathematical tasks. In this case, it was evident that the use of the dynamic software offered advantages to identify and explore initial relations empirically, later, the use of Derive software helped the participants explore the same relations algebraically.

In short, during the modelling processes, the participants had an opportunity of identifying and discussing assumptions and essential components that were relevant to construct a dynamic model of the task. The exploration of the model, through an inquisitive approach, led them to formulate a set of conjectures and relations that were important during the solution process. To reach the solution, they relied on empirical, numeric, visual, and algebraic approaches to support and validate conjectures. In this context, the use of the tool seems to help the participants experience diverse routes themselves to reconstruct basic mathematical results. In addition, the use of distinct computational tools became important for them to promote an inquisitive approach to the problems. These routes were key ingredients for teachers to identify instructional strategies that can foster their students’ development of mathematical thinking.

References


